QMUL-PH-2008-05

Aspects of Multiple Membranes

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Abstract

This paper examines various aspects of the recently proposed theory of coincident membranes by Bagger and Lambert. These include the properties of open membranes and the resulting boundary theory with an interpretation in terms of the fivebrane and marginal supersymmetric deformations of the interactions with the relation to the holographic dual.

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1 Introduction

Two of the most outstanding problems in M-theory are the understanding of the world volume theories of coincident membranes and coincident fivebranes [1]. In a fascinating recent development Bagger and Lambert [2–4] have proposed a Lagrangian description of coincident interacting membranes which relies on the scalar fields of the membrane world volume taking values in some non-associative algebra. (A similar approach was also described by Gustavsson in [5]). Remarkably, the theory could be made to possess the $\mathcal{N} = 8$ supersymmetry required of a membrane with the introduction of a *gauge* connection whose kinetic piece is a twisted version of Chern-Simons theory. The goal of this paper is to investigate properties of this theory. The two aspects will both involve breaking the supersymmetry in interesting ways. The first, through introduction of a boundary and the second, by directly altering the interaction potential while preserving some supersymmetry.

First, we will consider the theory of open membranes i.e. the Bagger Lambert theory with a boundary. The motivation for this is to see aspects of fivebrane physics from the point of view of the membrane boundary. If the Bagger Lambert action is a good description of the membrane then this would be expected since the fivebrane is the M-theory analogue of a D-brane.

The key issue is to examine the twisted Chern-Simons theory since this gives rise to dynamical degrees of freedom on the membrane boundary. In fact we will show that the membrane boundary is described by a sigma model whose target space is six dimensional. This is consistent with a self-dual string in the fivebrane world volume.

We will then consider possible marginal deformations of the interactions that preserve some fraction of supersymmetry. This is the M-theory analogue of the Leigh Strassler deformed $\mathcal{N} = 4$ Yang-Mills theory whose supergravity dual was described in [7]. We will examine the relationship between the proposed supersymmetry preserving deformations of the membrane theory and the deformed supergravity dual described and investigated in [8].

Whether this is the true description of coincident membranes is still not cer-

tain however supersymmetric field theories have proved of immense interest over the years [9] and has been shown to possess a rich and deep theoretical structure. Therefore the arrival of an entirely new supersymmetric field theory where the fields are nonassociative demands study.

2 An effective theory of interacting membranes

In [2] Bagger and Lambert proposed a theory with $\mathcal{N} = 8$ supersymmetry to describe multiple coincident membranes. The novel insight allowing this construction is that the fields take values in a nonassociative algebra, denoted here by $\mathcal{A}$. This nonassociative algebra, also called a three algebra, is endowed with a bilinear product and a totally antisymmetric three-bracket instead of the standard commutator found in Lie algebras. The three bracket or triple product is given by the antisymmetrised associator. For example the associator of three transverse scalars is

$$\langle X^I, X^J, X^K \rangle = (X^I \cdot X^J) \cdot X^K - X^I \cdot (X^J \cdot X^K) \tag{1}$$

and the three bracket is then

$$[X^I, X^J, X^K] = \frac{1}{12} \langle X^{[I}, X^J, X^{K]} \rangle \,. \tag{2}$$

One can introduce a basis $\{T^a\}$ of $\mathcal{A}$ satisfying

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d \,, \tag{3}$$

where the totally antisymmetric structure constants[4] f^{abcd} obey the fundamental identity, akin to the Jacobi identity of Lie algebras, given by

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d \,. \tag{4}$$

We remark that at this stage we have not specified the dimension of the algebra which we shall denote by n.

[4]We raise and lower algebraic indices with a positive definite trace form metric which, in this paper, we take to be simply δ_{ab}.

To make the supersymmetry algebra close [3] it is necessary to introduce non-propagating fields $\tilde{A}_\mu{}^b{}_a$, which gauge the transformation:

$$\delta X^I_a = \Lambda_{cd} f^{cdb}{}_a X^I_b \equiv \tilde{\Lambda}^b{}_a X^I_b \,. \tag{5}$$

The gauge field is antisymmetric as a consequence of the antisymmetry of $f^{cda}{}_b$ so the gauge group $G \subseteq SO(n)$. In fact since we are only examining the connection, one can only make statements about the algebra but it will be assumed that there is a full group structure.

As a consequence of the transformation law (5) the group G is restricted by insisting that one may write:

$$\tilde{A}_\mu{}^b{}_a = f^{cdb}{}_a A_{\mu\, cd} \tag{6}$$

for some $n \times n$ matrix valued $A_{\mu\, cd}$ with $f^{cdb}{}_a$ satisfying the fundamental identity which implies f^{abcd} must be an invariant four form of the group.

The Lagrangian for the full $\mathcal{N} = 8$ theory including these gauge fields is given by

$$\begin{aligned} \mathcal{L} \;&=\; -\frac{1}{2} D^\mu X^{aI} D_\mu X^I_a + \frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a + \frac{i}{4}\bar{\Psi}_b \Gamma_{IJ} \Psi_a X^I_c X^J_d f^{abcd} \\ &\quad -V(X) + \frac{1}{2}\epsilon^{\mu\nu\lambda}\left(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \right) , \end{aligned} \tag{7}$$

with bosonic potential

$$V(X) = \frac{1}{12} Tr\left([X^I, X^J, X^K]^2 \right) , \tag{8}$$

and supersymmetry transformations

$$\delta X^I_a = i\bar{\epsilon}\Gamma^I \Psi_a \,, \tag{9}$$

$$\delta \Psi_a = D_\mu X^I_a \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK} \epsilon \,, \tag{10}$$

$$\delta \tilde{A}_\mu{}^b{}_a = i\bar{\epsilon}\Gamma_\mu \Gamma^I X^I_c \Psi_d f^{cdb}{}_a \,, \tag{11}$$

where the covariant derivative acts as $D_\mu X_a = \partial_\mu X_a - \tilde{A}_\mu{}^b{}_a X_b$. The gauge kinetic term is similar to Chern-Simons theory but twisted with the structure constants of

the algebra. Obviously, the gauge fields are non-propagating as is required to give the correct degrees of freedom for supersymmetry.

In the remains of this paper we shall explore two truncations of this theory. We shall examine the gauge sector of the theory in closer detail by switching off the scalars and spinors. This will allow us to investigate the boundary theory of open coincident membranes. We shall then restrict our attention to the scalar-spinor sector and study marginal deformations of the theory that preserve $\mathcal{N} = 2$ supersymmetry.

3 Twisted Chern-Simons and the self-dual string boundary theory

When open membranes end on a fivebrane the boundary may be described by a $\mathcal{N} = (4,4)$ self-dual string theory (see for example, [1]). The self-dual string can be regarded from two perspectives; it can be viewed simply as a solitonic solution of the fivebrane world volume equations of motion [10] or one can think of it as the boundary theory of coincident membranes [11, 12].

From the fivebrane perspective one can take a *Maldacena style* limit and consider the resulting geometry of the fivebrane to describe the string. One finds that in this *near horizon* limit the self-dual string is described by a fivebrane with $AdS_3 \times S^3$ geometry [6].

In this section we shall study the self-dual string from the point of view of the boundary theory of coincident membranes. We will examine the gauge sector in isolation (setting $X^I = \Psi = 0$) and compare the resulting boundary theory to the fivebrane description.

To make concrete progress we need to specify a gauge group of the twisted Chern-Simons theory. This essentially means solving the fundamental identity (4). There is only one finite dimensional known solution (though other more exotic solutions have been discussed in [13]). When the dimension of the algebra is four it is known that

$$f^{abcd} \propto \epsilon^{abcd} \tag{12}$$

satisfies the fundamental identity and the associated gauge group is then $SO(4)$. There

is some evidence in the literature that this $n = 4$ algebra, which we denote by $\mathcal{A}_4$, may be the only possible solution to the fundamental identity [13, 14]. This restriction to the dimension four algebra is not something that we will find problematic, rather we find that this will have a natural interpretation in what follows. We will thus restrict ourself to the gauge group $G = SO(4)$.

First we carry out the usual decomposition of $SO(4) \cong SU(2) \times SU(2)$ by splitting the $so(4)$ gauge field into self-dual and anti-self-dual parts using 't Hooft matrices [15]. Let us perform this decomposition in our twisted Chern-Simons theory. We write the gauge field as

$$\tilde{A} = A^+ + A^- \,, \tag{13}$$

where

$$* A^+ = A^+ \,, \; *A^- = -A^- \tag{14}$$

and where $*$ denotes Hodge star on the matrix indices i.e.

$$(*A^+)_{ab} = \frac{1}{2}\epsilon^{abcd} A^+_{cd} \,. \tag{15}$$

We also have that $*^2 = 1$ and that for $\mathcal{A}_4$ the structure constants when viewed as operators have the same action on gauge fields as $*$. Using the relation (6) between $\tilde{A}$ and A we find (omitting indices for convenience) that

$$* \tilde{A} = A^+ - A^- = *fA = *^2 A = A \,. \tag{16}$$

Then the twisted Chern-Simons action becomes

$$\begin{aligned}
\mathcal{L}_{TCS} &= -\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right) \\
&= \frac{Tr}{4\pi}\left(Ad\tilde{A} + \frac{2}{3}A \wedge \tilde{A} \wedge \tilde{A}\right) \\
&= \frac{Tr}{4\pi}\left(A^+dA^+ + \frac{2}{3}A^+ \wedge A^+ \wedge A^+\right) - \frac{Tr}{4\pi}\left(A^-dA^- + \frac{2}{3}A^- \wedge A^- \wedge A^-\right) \\
&= \mathcal{L}_{CS}[A^+] - \mathcal{L}_{CS}[A^-] \,.
\end{aligned} \tag{17}$$

Implicitly we are working at level $k = 1$ and henceforward we shall drop the overall 4π normalisation of the Lagrangian. We see that the twisted Chern-Simons action has decomposed into two $SU(2)$ Chern-Simons theories but note that there is an all important relative minus sign between the two[5].

Let us now consider what happens when our membrane has a boundary. With a boundary, regular Chern-Simons theory gives rise to a (chiral) Wess-Zumino-Witten theory of propagating boundary degrees of freedom [16–18] (see also [19] for a helpful review). The naive first thought is that the boundary theory is a WZW model with a target space $S^3 \times S^3$. However the relative minus sign between the two Chern-Simons terms means we must be much more careful.

We shall briefly outline the process to arrive at the WZW model and see how this minus sign plays a role. We start by considering a single $SU(2)$ Chern-Simons theory with level $k = 1$. One writes the three manifold as $M = \Sigma \times \mathbf{R}$ with boundary $\partial M = \partial\Sigma \times \mathbf{R}$. For simplicity we consider only the case where Σ is a disc such that the boundary of M is a cylinder with angular coordinate θ and $\mathbf{R}$ identified with time. Upon doing a gauge variation of the Chern-Simons Lagrangian one picks up a surface term. To eliminate this we choose a suitable boundary condition, for example that the time component of the gauge field A_0 vanishes at the boundary. Equipped with this boundary condition one is able to recast the Chern-Simons theory as

$$S = \int d^3\sigma \epsilon^{0ij} \mathrm{Tr}\left(A_0 F_{ij} - A_i \partial_0 A_j\right) , \tag{18}$$

so that in the bulk A_0 becomes a Lagrange multiplier enforcing Gauss' law for the spatial components of the gauge field. Integrating out this Lagrange multiplier places the remaining components of the gauge field to be pure gauge $A_i = g^{-1}\partial_i g$. Writing out the resulting action in terms of the group element one finds

$$S_{WZW} = -\int_{\partial M} d\theta dt \mathrm{Tr}\left(g^{-1}\partial_\theta g g^{-1}\partial_t g\right) + \frac{1}{3}\int_M d^3\sigma \epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g\right) . \tag{19}$$

[5] Carrying out the same analysis but starting with the un-tilded A leads to an apparent relative minus sign on the A^- cubic term but this is not physically significant since it can be removed by a field redefinition.

This action is invariant under $g \to L(\theta) g R(t)$. $L(\theta)$ corresponds to transformations which do not vanish at $t = \pm\infty$ and therefore states lie in representations of this symmetry. $R(t)$ is a gauge transformation which must fixed. Hence the action is actually a chiral Wess-Zumino-Witten theory [18].

Let us explicitly calculate the action by parameterising the group element with Euler angles in the form

$$g(\theta, t) = e^{iu(\theta,t)\sigma_2} e^{ix(\theta,t)\sigma_3} e^{iv(\theta,t)\sigma_2} \,, \tag{20}$$

with

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{21}$$

We find the kinetic term contributes

$$-\,\mathrm{Tr}\left(g^{-1}\partial_\theta g g^{-1}\partial_t g\right) = 2\left(\partial_\theta x \partial_t x + \partial_\theta u \partial_t u + \partial_\theta v \partial_t v + \cos 2x \left(\partial_\theta u \partial_t v + \partial_\theta v \partial_t u\right)\right) . \tag{22}$$

The Wess-Zumino term is simply the pull back of the canonical three form of the group which we calculate as

$$\begin{aligned} \chi &= \frac{1}{3} Tr(g^{-1}dg)^3 \\ &= -4 \sin 2x \, dx \wedge du \wedge dv \\ &= d\left(2\cos 2x \, du \wedge dv\right) . \end{aligned} \tag{23}$$

Altogether we find the WZW action is

$$S_{WZW} = 2 \int d\theta dt \left(\partial_\theta x \partial_t x + \partial_\theta u \partial_t u + \partial_\theta v \partial_t v + 2\cos 2x \, \partial_\theta u \partial_t v\right) . \tag{24}$$

To see this in a more familiar form we do a field redefinition

$$u = \frac{1}{2}(\tau + \phi) \,, \quad v = \frac{1}{2}(\tau - \phi) \,, \tag{25}$$

and find after some trivial trigonometry

$$S_{WZW} = \int d\theta dt \left(\cos^2 x \, \partial_\theta \tau \partial_t \tau + \sin^2 x \, \partial_\theta \phi \partial_t \phi + \partial_\theta x \partial_t x + \cos 2x \left(\partial_\theta \tau \partial_t \phi - \partial_\theta \phi \partial_t \tau\right)\right) . \tag{26}$$

We can now read off the metric, the 2-form and its field strength by picking out the symmetric and antisymmetric terms in this action

$$\begin{aligned} ds^2 &= \cos^2 x\, d\tau^2 + \sin^2 x\, d\phi^2 + dx^2\,, && (27)\\ B &= \cos 2x\, d\phi \wedge d\tau\,, && (28)\\ H = dB &= 2\sin 2x\, dx \wedge d\phi \wedge d\tau = 4\sqrt{g} dx \wedge d\phi \wedge d\tau\,. && (29)\end{aligned}$$

The target space is a three-sphere (written in Hopf coordinates) with unit radius and with one unit of normalized flux through the sphere.

For the multiple M2 theory we can read off the complete six dimensional target space metric and field strength by simply taking into account the extra minus sign entering into (17) to find

$$\begin{aligned} ds^2 &= \cos^2 x\, d\tau^2 + \sin^2 x\, d\phi^2 + dx^2 - \cos^2 \tilde{x}\, d\tilde{\tau}^2 - \sin^2 \tilde{x}\, d\tilde{\phi}^2 - d\tilde{x}^2 && (30)\\ H = dB &= 2\sin 2x\, dx \wedge d\phi \wedge d\tau - 2\sin 2\tilde{x}\, d\tilde{x} \wedge d\tilde{\phi} \wedge d\tilde{\tau}\,. && (31)\end{aligned}$$

The flux is anti-self-dual i.e. $*H = -H$ and the geometry here can be thought of as a sphere of unit radius cross a sphere of imaginary (unit) radius. The sphere of imaginary radius is a space of constant negative curvature which we identify with AdS_3. Indeed, if one analytically continues $\tilde{x} \to i\tilde{x}$ so as to absorb the minus signs into the metric of the target space then the result would be the metric and flux for $AdS_3 \times S^3$ [6]. This is very satisfactory, from examining the boundary theory we see the target space is six dimensional with an anti-self-dual three form flux and a geometry of $AdS_3 \times S^3$. Comparing this to the fivebrane we see that this is indeed the description of the self-dual string in a *Maldacena style* limit [6]. The restriction to $SO(4)$ now makes sense since it ensures the dimension of the boundary theory target space is six i.e. the dimension of the fivebrane.

One may wonder whether one could reinstate k, the level, and introduce a parameter to control the number of membranes in this way. This has one appealing property

[6] This appears to be legitimate locally but one should be very careful in how to view the global nature of the space.

that the self-dual string charge will be given by k. However, there are various difficulties with this approach. First, the supersymmetry requires that we also multiply the matter sector by k which does not feel appropriate for the interacting membrane theory. Second, the scaling of radius of the target space S^3 with k does not match the scaling found in [6].

4 Marginal deformations

We now shift our attention back to closed membranes and begin by describing the Bagger Lambert theory in $\mathcal{N}=2$ superspace language. The gauge field is removed and so the $\mathcal{N}=8$ supersymmetry is no longer manifest and only an $SU(4)\times U(1)\subset SO(8)$ R-symmetry remains. The fields are redefined using a notation suited to the $SU(4)\times U(1)$ symmetry as follows:

$$\begin{array}{rclcc} X^I & \rightarrow & Z^A \oplus Z_{\bar{A}} & \in & \mathbf{4}(1)\oplus\bar{\mathbf{4}}(-1), \\ \Psi & \rightarrow & \psi^A\oplus\psi_{\bar{A}} & \in & \mathbf{4}(-1)\oplus\bar{\mathbf{4}}(1), \\ \epsilon & \rightarrow & \varepsilon\oplus\varepsilon^*\oplus\varepsilon^{AB} & \in & \mathbf{1}(-2)\oplus\bar{\mathbf{1}}(2)\oplus\mathbf{6}(0). \end{array} \tag{32}$$

The $\mathcal{N}=2$ supervariation in this form is,

$$\begin{aligned} \delta Z^A &= i\bar{\varepsilon}\psi^A, \\ \delta\psi^A &= 2\gamma^\mu\partial_\mu Z^A\varepsilon + i\kappa_1\epsilon^{ABCD}\big[Z_{\bar{B}},Z_{\bar{C}},Z_{\bar{D}}\big]\varepsilon^* + 3i\kappa_3\big[Z^A,Z^B,Z_{\bar{B}}\big]\epsilon\,. \end{aligned} \tag{33}$$

One may then formulate the theory in terms of $\mathcal{N}=2$ chiral superfields, $\mathcal{Z}^A$, which satisfy

$$\bar{D}\mathcal{Z}^A = 0, \tag{34}$$

with an expansion

$$\mathcal{Z}^A = Z^A(y) + \bar{\theta}^*\psi^A(y) + \bar{\theta}^*\theta F^A(y), \tag{35}$$

where $y^\mu = x^\mu + i\bar{\theta}\gamma^\mu\theta$. The Clifford algebra has a real basis and conjugation of the complex spinors ε and θ is defined as $\bar{\theta}=\theta^{*T}\gamma^0$.

The $\mathcal{N}=2$ supersymmetry algebra closes when the following constraints are imposed:

$$\begin{aligned}
\left[Z^A, Z^B, Z_{\bar{B}}\right] &= 0\,, & (36)\\
\left[Z^A, Z^B, \psi_{\bar{B}}\right] &= 0\,, & (37)\\
\left[\psi^A, Z^B, Z_{\bar{B}}\right] + \left[Z^A, \psi^B, Z_{\bar{B}}\right] &= 0\,. & (38)
\end{aligned}$$

These follow from the single superspace constraint

$$\left[\mathcal{Z}^A, \mathcal{Z}^B, \mathcal{Z}_{\bar{B}}\right] = 0 \tag{39}$$

and result in the vanishing of the third term in the supervariation of the fermion. Hence, in what follows, κ_3 does not appear in our analysis. When the dust settles one is left with the following Lagrangian written in terms of $\mathcal{N}=2$ chiral superfields obeying the constraint (39):

$$\mathcal{L} = \frac{1}{2}\int \mathrm{d}^4\theta\ \mathrm{Tr}\big(\mathcal{Z}^A, \mathcal{Z}_{\bar{A}}\big) + \int \mathrm{d}^2\theta\ W\big(\mathcal{Z}^A\big) + \int \mathrm{d}^2\theta^*\ \bar{W}\big(\mathcal{Z}_{\bar{A}}\big) \tag{40}$$

with W a holomorphic function of the non-associative algebra which describes the interaction:

$$W = -\frac{\kappa_1}{8}\epsilon_{ABCD}\mathrm{Tr}\big(\mathcal{Z}^A, \left[\mathcal{Z}^B, \mathcal{Z}^C, \mathcal{Z}^D\right]\big)\,. \tag{41}$$

In fact, closure of the full $\mathcal{N}=8$ theory requires $\kappa_1 = -\frac{1}{6}$ [3].

5 A Proposal for a Marginal Deformation

In [7, 8], the supergravity dual description of multiple membranes was studied and possible deformations preserving the AdS_4 structure were investigated. The preservation of the AdS_4 structure indicates the possibility of deforming the membrane theory while preserving conformal invariance. The deformation of the S^7 of course indicates that the deformation of the theory wouldn't preserve the $SO(8)$ R-symmetry and hence the $\mathcal{N}=8$ supersymmetry. This is the M-theory analogue of the Leigh Strassler deformation of $\mathcal{N}=4$ Yang-Mills whose supergravity dual was determined by Lunin and Maldacenca [7].

We will proceed by describing properties of the Leigh Strassler deformation and then describe the proposed M-theory analogue for the membrane theory. The result will be to allow a marginal deformation that preserves the $\mathcal{N}=2$ structure.

The Leigh Strassler deformation works by introducing a deformed product in the field theory parametrised by β (for this reason it is also referred to as β deformed Yang-Mills):

$$\begin{aligned} f * g &= e^{i\pi\beta\left(Q^1_f Q^2_g - Q^1_g Q^2_f\right)} f \cdot g \\ &= e^{i\pi\beta Q^1_{[f,} Q^2_{g]}} f \cdot g \,, \end{aligned} \tag{42}$$

where Q^i_f, Q^j_g denote the charges of the fields with respect to two global U(1) symmetries. As such the deformation picks out two U(1)'s as a special subgroup of the SO(6) R-symmetry.

This deformation only produces an effect within the superpotential. Expressed in $\mathcal{N}=1$ superspace using chiral fields $\Phi^I,\ I=1,2,3$ the superpotential is

$$W \sim \epsilon^{IJK} \Phi^I \Phi^J \Phi^K \,. \tag{43}$$

After deforming this using (42) the superpotential becomes

$$\Phi^1 \left[\Phi^2, \Phi^3\right]_* = \Phi^1 \left(e^{i\pi\beta}\Phi^2\Phi^3 - e^{-i\pi\beta}\Phi^3\Phi^2\right) . \tag{44}$$

Properties of the deformation can be summarised as follows;

1. The deformed product relies on the use of two U(1)'s.
2. The phase factor is defined in terms of the $U(1)_1 \times U(1)_2$ charges of the fields and involves a commutator over the field indices.
3. All products involving two different fields pick up a phase factor but effectively only commutators of independent fields are modified by the deformation.
4. The star product with any third field produces no new phases, for example

$$\Phi^1 * \left(\Phi^2 * \Phi^3\right) = \Phi^1 \cdot \left(\Phi^2 * \Phi^3\right) . \tag{45}$$

One should also note that the deformed product is very reminiscent of the noncommutative Moyal product and indeed from the D-brane perspective it is as if the transverse space to the brane has been made noncommutative. This analogy becomes more explicit from the perspective of the supergravity dual where one begins with the nondeformed brane solutions and then carries out a series of solution generating transformations to switch on background fields. One can then make the choice of whether the background fields are on the brane world volume, in which case the resulting deformation is to make the theory noncommutative, or transverse to the brane world volume in which case one produces this Leigh Strassler deformation.

We will now attempt now to generalise this deformation to the Bagger and Lambert theory by introducing a deformed triple product with analogous properties to the deformed Lie bracket. We introduce a phase factor in the triple product which depends on the charges of the fields under the $U(1)_1 \times U(1)_2 \times U(1)_3$ global symmetry. For chiral fields labelled A, B, C we propose the following deformed associator

$$< A, B, C >_* = e^{i\pi\beta Q^1_{[A,} Q^2_{B,} Q^3_{C]}} < A, B, C > \,, \tag{46}$$

where the Q^i denote the charges of the field under the $U(1)_1 \times U(1)_2 \times U(1)_3$ global symmetry. For all A, B, C this gives the phase factor

$$\begin{aligned} Q^1_{[A,} Q^2_{B,} Q^3_{C]} &= Q^1_A Q^2_B Q^3_C + Q^1_B Q^2_C Q^3_A + Q^1_C Q^2_A Q^3_B \\ &- Q^1_C Q^2_B Q^3_A - Q^1_B Q^2_A Q^3_C - Q^1_A Q^2_C Q^3_B \qquad (47) \\ &= \mathrm{Det}\mathcal{Q}\,, \qquad (48) \end{aligned}$$

where $\mathcal{Q}$ is the matrix of charges denoted by

$$\mathcal{Q} = \begin{pmatrix} Q^1_A & Q^2_A & Q^3_A \\ Q^1_B & Q^2_B & Q^3_B \\ Q^1_C & Q^2_C & Q^3_C \end{pmatrix}. \tag{49}$$

From this anti-symmetrisation it can be seen that cyclic permutations of $< A, B, C >$ will deform with the same phase structure as will cyclic permutations of $< C, B, A >$. Furthermore, the multiplicative phase factor accompanying the latter will be the reciprocal of the former. This deformation of products of three fields results in many

nice features, which are analogous to the Leigh Strassler deformation. In particular, the three bracket of the nonassociative algebra deforms in a manner analogous to the deformation of the commutator, that is:

$$\begin{aligned} [\mathcal{Z}_A, \mathcal{Z}_B, \mathcal{Z}_C]_* &= e^{i\pi\beta Q^1_{[A}, Q^2_{B}, Q^3_{C]}} \big(< \mathcal{Z}_A, \mathcal{Z}_B, \mathcal{Z}_C > \; + \text{cyclic} \big) \\ &- e^{-i\pi\beta Q^1_{[A}, Q^2_{B}, Q^3_{C]}} \big(< \mathcal{Z}_C, \mathcal{Z}_B, \mathcal{Z}_A > \; + \text{cyclic} \big), \end{aligned} \tag{50}$$

for all values of A, B, C. We find that properties of the deformation can be summarised as follows:

1. The deformed product relies on the use of three U(1)'s.

2. The phase factor is defined in terms of the $U(1)_1 \times U(1)_2 \times U(1)_3$ charges of the fields and involves a triple product over the field indices.

3. All products involving three different fields pick up a phase factor but effectively only triple products of independent fields are modified by the deformation.

4. The star product with any fourth field produces no new phases.

We find preservation of the algebraic properties of the theory. Furthermore, the reality condition is respected by the deformed triple product. For the purpose of explicit computation, the chiral fields have been assigned the following charges

$$\big(\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4\big) \;\to\; \big(\mathcal{Z}^1, \mathcal{Z}^2, e^{-i\varphi_1}\mathcal{Z}^3, e^{i\varphi_1}\mathcal{Z}^4\big) : \; U(1)_1, \tag{51}$$

$$\to\; \big(\mathcal{Z}^1, e^{-i\varphi_2}\mathcal{Z}^2, e^{i\varphi_2}\mathcal{Z}^3, \mathcal{Z}^4\big) : \; U(1)_2, \tag{52}$$

$$\to\; \big(e^{i\varphi_3}\mathcal{Z}^1, e^{-i\varphi_3}\mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4\big) : \; U(1)_3. \tag{53}$$

For all possible choices of $A \neq B \neq C \in (1,2,3,4)$ preserving the ordering in even permutations of $(1,2,3,4)$ the deformation is given by

$$\begin{aligned} \big[\mathcal{Z}^A, \mathcal{Z}^B, \mathcal{Z}^C\big]_* &= e^{i\pi\beta}\big(< \mathcal{Z}^A, \mathcal{Z}^B, \mathcal{Z}^C > + \text{cyclic} \big) \\ &- e^{-i\pi\beta}\big(< \mathcal{Z}^C, \mathcal{Z}^B, \mathcal{Z}^A > + \text{cyclic} \big), \end{aligned} \tag{54}$$

provided the sequence (A,B,C,D) in

$$\mathrm{Tr}\big(\mathcal{Z}^A,\big[\mathcal{Z}^B,\mathcal{Z}^C,\mathcal{Z}^D\big]_*\big), \tag{55}$$

can be written as a positive permutation of $(1,2,3,4)$. The cyclicity of the trace and the presence of a totally antisymmetric tensor in the superpotential ensure this can be done for every term in the action. This ordering is important for re-expressing the deformation simply. It can then be seen that the deformation preserves the properties of the associator within the trace

$$\mathrm{Tr}\big(\mathcal{Z}^A,<\mathcal{Z}^B,\mathcal{Z}^C,\mathcal{Z}^D>_*\big)=\mathrm{Tr}\big(<\mathcal{Z}^B,\mathcal{Z}^C,Z^D>_*,\mathcal{Z}^A\big)\,, \tag{56}$$

for all inequivalent $A,B,C,D\in(1,2,3,4)$ and that using this property we can then show

$$\mathrm{Tr}\big(\mathcal{Z}^A,\big\langle\mathcal{Z}^B,\mathcal{Z}^C,\mathcal{Z}^D\big\rangle_*\big)=-\mathrm{Tr}\big(\big\langle\mathcal{Z}^A,\mathcal{Z}^B,\mathcal{Z}^C\big\rangle_*,\mathcal{Z}^D\big)\,. \tag{57}$$

It then follows for the triple product

$$\mathrm{Tr}\big(\mathcal{Z}^A,\big[\mathcal{Z}^B,\mathcal{Z}^C,\mathcal{Z}^D\big]_*\big)=-\mathrm{Tr}\big(\big[\mathcal{Z}^A,\mathcal{Z}^B,\mathcal{Z}^C\big]_*,\mathcal{Z}^D\big)\,. \tag{58}$$

Analysis of anti-chiral fields reveals the same results with all exponential factors mapping to their reciprocal. Also note that, triple products containing triple products obey:

$$\big[\alpha,\beta,\big[A,B,C\big]_*\big]_*=\big[\alpha,\beta,\big[A,B,C\big]_*\big]\,. \tag{59}$$

As for $\mathcal{N}=4$ super Yang-Mills, the effects of the deformation on the action are found only in the superpotential. The $\mathcal{N}=2$ Lagrangian becomes

$$\mathcal{L}_*=\frac{1}{2}\int\mathrm{d}^4\theta\;\mathrm{Tr}\big(\mathcal{Z}^A,\mathcal{Z}_{\bar{A}}\big)-\frac{\kappa_1}{8}\epsilon_{ABCD}\int\mathrm{d}^2\theta\mathrm{Tr}\big(\mathcal{Z}^A,\big[\mathcal{Z}^B,\mathcal{Z}^C,\mathcal{Z}^C\big]_*\big)+\int\mathrm{d}^2\theta^*\bar{\mathcal{W}}_*(\mathcal{Z}_{\bar{A}}). \tag{60}$$

The supersymmetry algebra remains unchanged along with the superspace constraint

$$\big[\mathcal{Z}^A,\mathcal{Z}^B,\mathcal{Z}_{\bar{B}}\big]=0\,. \tag{61}$$

That the deformed theory can be expressed in $\mathcal{N}=2$ superspace means that $\mathcal{N}=2$ supersymmetry is manifest. In fact one may have preserved more but we expect only one quarter of the supersymmetry to be preserved as in Leigh Strassler.

Note that although we began with the $\mathcal{N}=2$ version of the multi-membrane theory, we may consider this as a deformation to the full $\mathcal{N}=8$ theory that preserves $\mathcal{N}=2$ supersymmetry. The route via the $\mathcal{N}=2$ superspace version of the membrane was chosen because of familiarity with the β-deformation of $\mathcal{N}=4$ SYM theory and as in that case is much easier to formulate using the superfield language.

We would like to see the effect of the deformation on the bosonic scalar potential:

$$V=\frac{1}{2\cdot 3!}\,\mathrm{Tr}\big(\big[X^I,X^J,X^K\big],\big[X^I,X^J,X^K\big]\big). \tag{62}$$

To start with we will look at the terms arising in the undeformed superpotential that survive the Grassman integration. There are four different types of terms that do this. These are $\big(Z,[Z,\psi,\psi]\big)$, $\big(\psi,[Z,Z,\psi]\big)$, $\big(F,[Z,Z,Z]\big)$ and $\big(Z,[Z,Z,F]\big)$. We neglect the fermions for the moment. Symmetries of the triple-product and the properties of the trace allow us to group terms

$$\begin{aligned}
\mathcal{W}|_{\theta\theta} \;=\; & -\frac{\kappa_1}{8}\epsilon^{ABCD}\mathrm{Tr}\big(Z^A,\big[Z^B,Z^C,\epsilon^{DEFG}\big[Z_{\bar E},Z_{\bar F},Z_{\bar G}\big]\big]\big)\\
& -\frac{\kappa_1}{8}\epsilon^{ABCD}\mathrm{Tr}\big(Z^A,\big[Z^B,\epsilon^{CEFG}\big[Z_{\bar E},Z_{\bar F},Z_{\bar G}\big],Z^D\big]\big)\\
& -\frac{\kappa_1}{8}\epsilon^{ABCD}\mathrm{Tr}\big(Z^A,\big[\epsilon^{BEFG}\big[Z_{\bar E},Z_{\bar F},Z_{\bar G}\big],Z^C,Z^D\big]\big)\\
& -\frac{\kappa_1}{8}\epsilon^{ABCD}\mathrm{Tr}\big(\epsilon^{AEFG}\big[Z_{\bar E},Z_{\bar F},Z_{\bar G}\big],Z^B,Z^C,Z^D\big]\big),
\end{aligned} \tag{63}$$

which becomes

$$\begin{aligned}
\mathcal{W}|_{\theta\theta} \;=\; & -\frac{24\kappa_1}{8}\mathrm{Tr}\big(Z^A,\big[\big[Z^C,Z^D,Z_{\bar C}\big],Z_{\bar D},Z_{\bar A}\big]\big)\\
& -\frac{24\kappa_1}{8}\mathrm{Tr}\big(Z^A,\big[Z_{\bar C},\big[Z^C,Z^D,Z_{\bar D}\big],Z_{\bar A}\big]\big)\\
& -\frac{24\kappa_1}{8}\mathrm{Tr}\big(Z^A,\big[Z_{\bar C},Z_{\bar D},\big[Z^C,Z^D,Z_{\bar A}\big]\big]\big).
\end{aligned} \tag{64}$$

The first and second terms vanish using the constraint

$$\big[Z^A,Z^B,Z_{\bar B}\big]=0, \tag{65}$$

whilst, using the symmetries of the trace and triple-product gives

$$
\begin{aligned}
\mathcal{W}|_{\theta\theta} &= -\frac{24\kappa_1}{8}\mathrm{Tr}\big(Z^A,[Z_{\bar{C}},Z_{\bar{D}},[Z^C,Z^D,Z_{\bar{A}}]]\big)\\
&= -\frac{24\kappa_1}{8}\mathrm{Tr}\Big(Z^A,\big([[Z_{\bar{C}},Z_D,Z^C],Z^D,Z_{\bar{A}}]\\
&\qquad +[Z^C,[Z_{\bar{C}},Z_{\bar{D}},Z^D],Z_{\bar{A}}]\\
&\qquad +[Z^C,Z^D,[Z_{\bar{C}},Z_{\bar{D}},Z_{\bar{A}}]]\big)\Big),\\
&= -\frac{24\kappa_1}{8}\mathrm{Tr}\big(Z^A,[Z^C,Z^D,[Z_{\bar{C}},Z_{\bar{D}},Z_{\bar{A}}]]\big),\\
&= \frac{24\kappa_1}{8}\mathrm{Tr}\big([Z^A,Z^B,Z^C],[Z_{\bar{A}},Z_{\bar{B}},Z_{\bar{C}}]\big).
\end{aligned} \tag{66}
$$

In the undeformed theory, this term is proportional to the term quadratic in the auxiliary field, which arises from the chiral superfield kinetic term. These combine to produce the sixth order scalar field term in the potential of the component Lagrangian.

The chiral kinetic term of the $\mathcal{N}=2$ superfield Lagrangian is invariant under the deformation but we have already seen that the triple-products, present in the superpotential, are modified while the constraint equations are not. The field equation for the auxiliary field becomes

$$
F^A = \kappa_1\epsilon^{ABCD}\big[Z_{\bar{B}},Z_{\bar{C}},Z_{\bar{D}}\big]_* . \tag{67}
$$

After the deformation, the superpotential can be written, using the deformed symmetries of the trace and triple product, as

$$
\mathcal{W}_* = -3\kappa_1\mathrm{Tr}\big(\mathcal{Z}^1,\big[\mathcal{Z}^2,\mathcal{Z}^3,\mathcal{Z}^4\big]_*\big) . \tag{68}
$$

That both the auxiliary field equation and the superpotential can be simply expressed in the deformed theory by replacing the triple product with a star-triple product is due to the symmetries of the star product outlined earlier.

As for the undeformed case, the only purely bosonic contribution coming from the superpotential is of the same form as the term coming from the chiral kinetic piece and is given by

$$
\mathcal{W}|_{\theta\theta} \sim \mathrm{Tr}\big(\big[Z^A,Z^B,Z^C\big]_*,\big[Z_{\bar{A}},Z_{\bar{B}},Z_{\bar{C}}\big]_*\big) . \tag{69}
$$

5.1 The β-Deformation related to the supergravity dual

We now wish to examine other deformations, which may be related to deformations of the $AdS_4 \times S^7$ supergravity dual of the membrane. Matching such deformations will provide further support to the Bagger Lambert conjectured multi-membrane theory.

The deformations in [8] preserve the AdS_4 (and hence the conformality of the membrane theory) but deform the S^7 using M-theory solution generating transformations. These solution generating transformations involve identifying a three torus and then acting with a solution generating transformation on a $T^2 \subset T^3$ to deform the solution. It was shown that different choices of the T^2 produced distinct solutions. A whole spectrum of solutions could be generated with a great diversity in their apparent properties. In one case the effects of the deformation were such that the entropy of related black-brane was left invariant by the deformation. In another case, the deformations produced a different entropy involving a simple multiplicative factor with the deformation parameter. There were also more complicated examples.

Motivated by the deformation of the supergravity dual, of a T^2, we propose the deformation of the membrane theory should be with a deformed two product. The deformed associator, given by replacing the regular product with the star product as given in (42), is

$$< A, B, C >_{*} = A * (B * C) - (A * B) * C \,. \tag{70}$$

For star products of the nonassociative theory we have no reason to assume that we will recover the properties that were found for the star product in $\mathcal{N} = 4$ SYM or those described in section 5. Even if these properties were preserved then we could not necessarily expect the two terms present in the associator to transform in an homogenous manner. It would then be impossible to express the deformed associator as a product of a phase (containing all the information of the deformation) and the undeformed associator as we did in (46). Such a factorisation was crucial in allowing us to express the deformed action simply.

Interestingly, for some choices of T^2 this factorisation property is present. For other

choices we lose this property and star-products with third fields produce additional phase-factors under the deformation. However, this can occur in such a way as to ensure that the two constituent terms in any associator transform in an homogenous manner. Again, we find we are able to generate families of solutions through the deformation with very different properties. All results depend on the particular choice of fields and charge assignments of the $U(1) \times U(1)$ subgroup. Let us see how the associator deforms under the star product. Using this deformation, the product of three fields always transforms like

$$A * (B * C) = e^{i\pi\beta Q^1_{[A,} Q^2_{BC]}} e^{i\pi\beta Q^1_{[B,} Q^2_{C]}} \; A \cdot (B \cdot C) \,, \tag{71}$$

where Q^i_{BC} is the charge of $(B \cdot C)$ under the global symmetry $U(1)_i$. Additivity of charge for this global symmetry gives the phase as

$$e^{i\pi\beta\left(Q^1_{[A,} Q^2_{B]} + Q^1_{[A,} Q^2_{C]} + Q^1_{[B,} Q^2_{C]}\right)} \,, \tag{72}$$

where the ordering of the field indices is determined by the ordering of the fields in the product. We can see now, that given three fields the constituent terms within the associator can be expected to transform in the same way under the deformation. Furthermore, we can conclude that associators related by complete anti-symmetry

$$< A, B, C > \longleftrightarrow < C, B, A > \tag{73}$$

will pick up reciprocal phase factors. However we lose the ability to relate phase factors for cyclic permutations of all associators. This is a result of the replacement of a product of three field charges, anti-symmetrised on the field indices, with three commutators of field charges. This symmetry is lost for deformations of general associators and triple products under arbitrary global $U(1) \times U(1)$'s.

We look now at two particular examples to highlight the behaviour of the non-associative Bagger Lambert theory under this deformation.

5.2 Example I: $\left[\mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4\right]$ under $U(1)_1 \times U(1)_3$

In this example, the star-product with a third field produces a new phase factor under the deformation. This happens in such a way as to ensure each associator transforms into the old undeformed associator and a multiplicative phase factor. Using the arbitrary charge assignments from before and deforming under $U(1)_1 \times U(1)_3$ we see

$$\begin{aligned}
\left[\mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4\right]_* &= <\mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4> - <\mathcal{Z}^4, \mathcal{Z}^3, \mathcal{Z}^2> \\
&+ <\mathcal{Z}^3, \mathcal{Z}^4, \mathcal{Z}^2> - <\mathcal{Z}^2, \mathcal{Z}^4, \mathcal{Z}^3> \\
&+ e^{-2\pi i\beta} <\mathcal{Z}^4, \mathcal{Z}^2, \mathcal{Z}^3> - e^{2\pi i\beta} <\mathcal{Z}^3, \mathcal{Z}^2\mathcal{Z}^4> . \qquad (74)
\end{aligned}$$

The triple products do not transform in a similar manner to that of the $U(1)\times U(1)\times U(1)$ deformation defined earlier. However, it may be relatable to one of the less symmetric cases found for the M2 supergravity dual.

5.3 Example II: $\left[\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3\right]$ under $U(1)_2 \times U(1)_3$

In this example, we find that star-products with a third field produce no new deformative phase factors just as we found in with the original β-deformation. The deformation can therefore be fully expressed in terms of a deformation of the first product taken within any associator. Furthermore, for this particular choice, each and every bi-linear star-product produces exactly the correct phase factor to reproduce the effects of the deformation defined using three global $U(1)$ symmetries. For two deformations with very different origins it is surprising that we find the same resultant deformation of triple-product terms within the superpotential. We find, under $U(1)_2 \times U(1)_3$,

$$\begin{aligned}
\left[\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3\right]_* &= <\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3>_* + <\mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^1>_* \\
&+ <\mathcal{Z}^3, \mathcal{Z}^1, \mathcal{Z}^2>_* - <\mathcal{Z}^3, \mathcal{Z}^2, \mathcal{Z}^1>_* \\
&- <\mathcal{Z}^2, \mathcal{Z}^1, \mathcal{Z}^3>_* - <\mathcal{Z}^1, \mathcal{Z}^3, \mathcal{Z}^2>_* . \qquad (75)
\end{aligned}$$

On further investigation it was found that third and higher order star products were irrelevant in the deformation

$$\begin{aligned}\left[\mathcal{Z}^1,\mathcal{Z}^2,\mathcal{Z}^3\right]_* &= \mathcal{Z}^1\big(\mathcal{Z}^2*\mathcal{Z}^3\big)-\big(\mathcal{Z}^1*\mathcal{Z}^2\big)\mathcal{Z}^3+\mathcal{Z}^2\big(\mathcal{Z}^3*\mathcal{Z}^1\big)\\ &- \big(\mathcal{Z}^2*\mathcal{Z}^3\big)\mathcal{Z}^1+\mathcal{Z}^3\big(\mathcal{Z}^1*\mathcal{Z}^2\big)-\big(\mathcal{Z}^3*\mathcal{Z}^1\big)\mathcal{Z}^2\\ &- \mathcal{Z}^3\big(\mathcal{Z}^2*\mathcal{Z}^2\big)+\big(\mathcal{Z}^3*\mathcal{Z}^2\big)\mathcal{Z}^1-\mathcal{Z}^2\big(\mathcal{Z}^1*\mathcal{Z}^3\big)\\ &+ \big(\mathcal{Z}^2*\mathcal{Z}^1\big)\mathcal{Z}^3-\mathcal{Z}^1\big(\mathcal{Z}^3*\mathcal{Z}^2\big)+\big(\mathcal{Z}^1*\mathcal{Z}^3\big)\mathcal{Z}^2. \end{aligned}\tag{76}$$

This can be written

$$\begin{aligned}\left[\mathcal{Z}^1,\mathcal{Z}^2,\mathcal{Z}^3\right]_* &= e^{i\pi\beta}\big(<\mathcal{Z}^1,\mathcal{Z}^2,\mathcal{Z}^3>+\text{cyclic}\big)\\ &- e^{-i\pi\beta}\big(<\mathcal{Z}^3,\mathcal{Z}^2,\mathcal{Z}^1>+\text{ cyclic}\big)\,. \end{aligned}\tag{77}$$

Furthermore, if we then take the product with $\mathcal{Z}^4$ from

$$\mathrm{Tr}\Big(\mathcal{Z}^4,\left[\mathcal{Z}^1,\mathcal{Z}^2,\mathcal{Z}^3\right]_*\Big)_*\tag{78}$$

we obtain

$$\mathrm{Tr}\Big(\mathcal{Z}^4,\left[\mathcal{Z}^1,\mathcal{Z}^2,\mathcal{Z}^3\right]_*\Big),\tag{79}$$

which produces exactly the same term that we would generate using the $U(1)_1 \times U(1)_2 \times U(1)_3$ deformation completely anti-symmetrised over the three field indices as already described in section 5. Thus the marginal deformation described in section 5 will be dual to the deformation described in [8].

6 Conclusions

We are still some way from a full understanding the interacting membrane theory but there are significant indications that we are heading in the right direction. One, we have reproduced aspects of the fivebrane from the open membrane. Two, we have explicit examples of supersymmetry preserving marginal deformations that may be related to deformations of the supergravity dual. The key issue that needs to be further explored is to understand how the number of membranes enters the theory

and of course relate this to the number of degrees of freedom and hopefully to the proposal described in [20].

An immediate technical question is the study of the membrane supersymmetry in the presence of a boundary and the supersymmetry of the boundary theory. One can also try to understand quantum properties of the twisted Chern-Simons theory. Famously, the partition function of Chern-Simons theory [17] leads to the Ray-Singer Torsion of the three manifold. The role of the Ray-Singer torsion of the membrane and the partition function of the Chern-Simons theory at level -1 (coming from the anti-self-dual sector) are still very much open questions.

Note Added:

As this paper was being prepared, two interesting preprints appeared discussing aspects of the Bagger and Lambert theory [21, 22].

7 Acknowledgements

This work was in part supported by the EC Marie Curie Research Training Network, MRTN-CT-2004-512194. LT and DT are supported by an STFC (nee PPARC) grant. We are gratefull to Tom Brown, Paul Heslop, Sanjaye Ramgoolam and Bill Spence for discussions.

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